



# Classical and Bayesian Estimation in Exponential Power Distribution under Type-I Progressive Hybrid Censoring with Binomial Removals

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## Abstract

This article deals with the classical and Bayesian estimation in exponential power distribution based on Type-I progressive hybrid censoring with binomial removals at each stage. Based on the considered censoring scheme, the maximum likelihood estimates and their coverage probabilities are computed by the Monte Carlo simulation technique. MCMC technique is used to obtain the Bayes estimates under the informative priors. The performance of both the approaches is evaluated in terms of their absolute bias and mean square error (MSE) as well as the width of the confidence interval. Applicability of the suggested approach is illustrated by analysis of a real-life dataset.

**Keywords:** Type-I progressive hybrid censoring; binomial removals; Monte Carlo simulation technique; MCMC technique; coverage probability.

# 1 Introduction

Type-I progressive hybrid censoring scheme (PHCS) was introduced by [8]. As the name, this censoring scheme is the combination of progressive and hybrid censoring schemes. In this censoring scheme, we terminate the experiment either failure a pre-fixed number of units or hitting a pre-decided time point whichever comes first by removing a number of survival units from the experiment at the time of each failure. [3], [1] and [12] estimated the parameters and drew the inference about the parameters of different lifetime distributions under this scheme. They assumed that the number of removals of the survival units from the experiment is fixed in advance.

There are many situations where we cannot prefix the number of removals in advanced. For example, suppose some critical heart disease patients are kept under treatment in a hospital. At the time of death of the first patient, some of them may leave the hospital assuming that the treatment is not proper and may shift to other hospitals. Similarly, at the time of the second death, some patients leave the hospital. The procedure of leaving the hospital may continue. In such situations, we cannot fix the number of removals. Therefore, we may treat the number of removals of the survival units from the test at different stages as a random variable instead as fixed. So, the number of removals can be treated as random. If we use the concept of random removals in PHCS then the censoring is called Type-I progressive hybrid censoring with random removals. [7] discussed the estimation for the exponential power distribution considering removals follow the binomial distribution. [15] obtained the Bayes estimate of competing risk model and [11] discussed the inference for a geometric-Poisson-Rayleigh distribution under Type-I PHCS with random removals.

This motivates us to study the estimation of the parameters of other useful distribution based on the Type-I PHCS with random removals. In this paper, we discuss the estimation of the parameters of the exponential power distribution (EPD) introduced by [13] which is one of very few natural two-parameter distributions has a U-shaped hazard function and different applications in real-life.

If lifetime ( $X$ ) of the units follows the EPD then the cumulative distribution function (CDF)  $F_X(x)$  and probability density function (PDF)  $f_X(x)$  of  $X$  are given by

$$F_X(x) = 1 - \exp(1 - e^{(\zeta x)^\delta}), \quad x > 0, \quad (1)$$

$$f_X(x) = \delta \zeta^\delta x^{\delta-1} e^{(\zeta x)^\delta} \exp(1 - e^{(\zeta x)^\delta}), \quad x > 0, \quad (2)$$

where  $\delta, \zeta > 0$  are shape and scale parameters respectively.

The rest of the article is as follows. In Section 2, we explain the implementation of Type-I PHCS with binomial removals. Sections 3 and 4 are devoted to theoretically derivation of the ML and Bayes estimators under the concerned censoring scheme, respectively. In Section 5, we conducted a simulation study for validate the results of Sections 3 and 4. Applicability of the suggested approach is illustrated by analysis of a real-life dataset in Section 6. We summarize the conclusions in Section 7.

# 2 Implementation of Type-I PHCS with Binomial Removals

Suppose  $n$  identical units whose lifetime follows EPD are put on a life-testing experiment. Also, assume that the number of maximum units  $\kappa$  ( $0 < \kappa < n$ ) for which we require the failure informa-

tion and maximum time  $\tau$  up to which we can continue the experiment are pre-fixed beforehand. At the time of the first failure, say  $X_{1:\kappa:n}$ , we randomly remove  $R_1$  ( $0 \leq R_1 \leq n - \kappa$ ) survival units from the experiment (from the remaining  $n - 1$  units) with the probability of removals being  $p$ . As the second failure occurs at the time, say  $X_{2:\kappa:n}$ , we randomly remove  $R_2$  ( $0 \leq R_2 \leq n - \kappa - R_1$ ) survival units from the experiment (from the remaining  $n - 2 - R_1$  units) with the same probability  $p$ . The experiment continues in the same way until the  $\kappa^{th}$  failure occurs or the pre-fixed time point  $\tau$ , whichever comes first. Therefore, under the Type-I PHCS with binomial removals, we terminate the experiment at the time point  $\tau^* = \min(X_{\kappa:\kappa:n}, \tau)$ . If the  $\kappa^{th}$  failure comes before  $\tau$ , then we remove all remaining survival units  $R_\kappa = n - \kappa - \sum_{i=1}^{\kappa-1} R_i$  from the experiment at  $X_{\kappa:\kappa:n}$  and we will terminate the experiment at that time. To ensure  $\kappa$  failures, we also assume that  $R_i$  should be less than  $n - \kappa - \sum_{j=1}^{i-1} R_j$ . This assumption does not change the essence of our experiment, as it is only the task to decide the number of removals. If the  $\kappa^{th}$  failure does not come before  $\tau$  and only  $\omega$  ( $0 < \omega < \kappa$ ) failures occur before  $\tau$  then at that time, say  $X_{\omega:\kappa:n}$ , we randomly remove  $R_\omega$  ( $0 \leq R_\omega \leq n - \kappa - \sum_{i=1}^{\omega-1} R_i$ ) units with the probability of removals  $p$  and we terminate the experiment at time point  $\tau$  by removing all survival units  $R_\omega^* = n - \omega - \sum_{i=1}^{\omega} R_i$  with probability 1 at that time point. It is also noted that no failure has occurred between  $X_{\omega:\kappa:n}$  and  $\tau$ .

Based on the above Type-I PHCS with binomial removals, we shall note the two forms of data as

- Case-1:  $\{X_{1:\kappa:n}, X_{2:\kappa:n}, \dots, X_{\kappa:\kappa:n}\}$  if  $\tau \geq X_{\kappa:\kappa:n}$ .
- Case-2:  $\{X_{1:\kappa:n}, X_{2:\kappa:n}, \dots, X_{\omega:\kappa:n}\}$  if  $\tau < X_{\kappa:\kappa:n}$  and  $X_{\omega:\kappa:n} < \tau < X_{\omega+1:\kappa:n}$ .

It is observed that if  $\tau^* = \tau$  then Type-I PHCS converts in simple progressive Type-I scheme and if  $\tau^* = X_{\kappa:\kappa:n}$  then it converts into the simple progressive Type-II.

Here, we remove the units with constant probability and independently, therefore, we can let that  $R_i$  follows a binomial distribution with parameter  $(n - \kappa - \sum_{j=1}^{i-1} r_j)$  and  $p$ , where  $i = 1, 2, \dots, \kappa - 1$  when  $\tau \geq X_{\kappa:\kappa:n}$  and  $i = 1, 2, \dots, \omega$  when  $\tau < X_{\kappa:\kappa:n}$ .

### 3 Maximum Likelihood Estimation

Let  $n$  identical units whose lifetime follows EPD be put on a life testing experiment. Also, let  $X_{1:\kappa:n} \leq X_{2:\kappa:n} \leq \dots \leq X_{\nu:\kappa:n}$  be the lifetimes of the units under Type-I PHCS with binomial removals  $(n, \kappa, R_1, R_2, \dots, R_\nu)$ , where

$$\nu = \begin{cases} \kappa & (0 < \kappa < n) \text{ if } \tau \geq X_{\kappa:\kappa:n} \\ \omega & (0 < \omega < \kappa) \text{ if } \tau < X_{\kappa:\kappa:n} \end{cases}.$$

The joint likelihood function of  $\delta$  and  $\zeta$  given  $\underline{X} = (X_{1:\kappa:n}, X_{2:\kappa:n}, \dots, X_{\nu:\kappa:n})$  can be written as

$$L(\delta, \zeta | \underline{x}, \underline{r}) = C^* \prod_{i=1}^{\nu} f_X(x_{i:\kappa:n}) [1 - F_X(x_{i:\kappa:n})]^{T_i} [1 - F_X(\tau^*)]^{T_\nu^*}, \quad (3)$$

where

$$C^* = \begin{cases} \prod_{i=1}^{\kappa} \left( n - \sum_{j=1}^{i-1} (r_j + 1) \right); & \tau \geq X_{\kappa:\kappa:n}, \\ \prod_{i=1}^{\omega} \left( n - \sum_{j=1}^i (r_j + 1) \right); & \tau < X_{\kappa:\kappa:n}, \end{cases}$$

$$r_{\nu}^* = \begin{cases} 0; & \tau \geq X_{\kappa:\kappa:n}, \\ n - \omega - \sum_{i=1}^{\omega} R_i; & \tau < X_{\kappa:\kappa:n}. \end{cases}$$

After putting the values of  $F_X(x)$  and  $f_X(x)$  from Equations (1) and (2) in Equation (3), we can get  $L(\delta, \zeta | \underline{x}, \underline{r})$  as

$$L(\delta, \zeta | \underline{x}, \underline{r}) = C^* \delta^{\nu} \zeta^{\nu \delta} \prod_{i=1}^{\nu} x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^{\delta}} \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^{\delta}} \right) \right\}^{r_i+1} \left\{ \exp \left( 1 - e^{(\zeta \tau^*)^{\delta}} \right) \right\}^{r_{\nu}^*}. \quad (4)$$

The PDF of  $R$  at each stage is written as

$$P[R_1 = r_1] = \binom{n-\kappa}{r_1} p^{r_1} (1-p)^{n-\kappa-r_1}; r_1 = 0, 1, 2, \dots, n-\kappa, \quad (5)$$

and for  $i = 2, 3, \dots, \kappa-1$ ,

$$P[R_i = r_i | R_{i-1} = r_{i-1}, \dots, R_1 = r_1] = \binom{n-\kappa - \sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} \\ \times (1-p)^{n-\kappa - \sum_{j=1}^i r_j}; r_i = 0, 1, 2, \dots, n-\kappa - \sum_{j=1}^{i-1} r_j. \quad (6)$$

Similarly,

$$P[R_{\omega} = r_{\omega} | R_{\omega-1} = r_{\omega-1}, \dots, R_1 = r_1] = \binom{n-\kappa - \sum_{j=1}^{\omega-1} r_j}{r_{\omega}} p^{r_{\omega}} \\ \times (1-p)^{n-\kappa - \sum_{j=1}^{\omega} r_j}; r_{\omega} = 0, 1, 2, \dots, n-\kappa - \sum_{j=1}^{\omega-1} r_j. \quad (7)$$

Since at the time of the  $\kappa^{th}$  failure or time point  $\tau$ , we remove all survival units from the experiment with probability one, therefore,

$$P[R_{\kappa} = r_{\kappa} | R_{\kappa-1} = r_{\kappa-1}, \dots, R_1 = r_1] = 1 = P[R_{\omega}^* = r_{\omega}^* | R_{\omega} = r_{\omega}, R_{\omega-1} = r_{\omega-1}, \dots, R_1 = r_1].$$

Thus, the joint PDF of  $\underline{R} = (R_1, R_2, \dots, R_{\nu})$  can be obtained as

$$P[\underline{R} = \underline{r} | p] = \begin{cases} P[R_1 = r_1] P[R_2 = r_2 | R_1 = r_1] \times \dots \\ \times P[R_{\kappa-1} = r_{\kappa-1} | R_{\kappa-2} = r_{\kappa-2}, \dots, R_1 = r_1]; & \tau \geq X_{\kappa:\kappa:n}, \\ P[R_1 = r_1] P[R_2 = r_2 | R_1 = r_1] \times \dots \\ \times P[R_{\omega} = r_{\omega} | R_{\omega-1} = r_{\omega-1}, \dots, R_1 = r_1] & ; \tau < X_{\kappa:\kappa:n}. \end{cases} \quad (8)$$

From Equations (5), (6), (7) and (8), we get the joint PDF of the number of removals units ( $R$ ) at each stage as

$$P \left[ \tilde{R} = \tilde{r} | p \right] = \begin{cases} \frac{(n-\kappa)!}{\left( n-\kappa-\sum_{j=1}^{\kappa-1} r_j \right)! \prod_{j=1}^{\kappa-1} r_j!} p^{\sum_{j=1}^{\kappa-1} r_j} (1-p)^{(\kappa-1)(n-\kappa)-\sum_{j=1}^{\kappa-1} (\kappa-j)r_j}; \\ \tau \geq X_{\kappa:\kappa:n}, \\ \frac{(n-\kappa)!}{\left( n-\kappa-\sum_{j=1}^{\omega} r_j \right)! \prod_{j=1}^{\omega} r_j!} p^{\sum_{j=1}^{\omega} r_j} (1-p)^{\omega(n-\kappa)-\sum_{j=1}^{\omega} (\omega+1-j)r_j}; \\ \tau < X_{\kappa:\kappa:n}. \end{cases} \quad (9)$$

Also, assume that  $X_{i:\kappa:n}$ 's do not depend on  $R_i$ 's. Therefore, the joint likelihood function of  $\delta, \zeta$  and  $p$  can be obtained as

$$L(\delta, \zeta, p | \underline{x}, \underline{r}) = C^{**} L_1(\delta, \zeta | \underline{x}, \underline{r}) \times L_2(p | \underline{r}),$$

where  $C^{**}$  is a constant and

$$L_1(\delta, \zeta | \underline{x}, \underline{r}) = \delta^\nu \zeta^{\nu\delta} \prod_{i=1}^{\nu} x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^\delta} \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \right\}^{r_i+1} \\ \times \left\{ \exp \left( 1 - e^{(\zeta \tau^*)^\delta} \right) \right\}^{r_\nu^*}. \quad (10)$$

$$L_2(p | \underline{r}) = \begin{cases} p^{\sum_{j=1}^{\kappa-1} r_j} (1-p)^{(\kappa-1)(n-\kappa)-\sum_{j=1}^{\kappa-1} (\kappa-j)r_j}; \tau \geq X_{\kappa:\kappa:n}, \\ p^{\sum_{j=1}^{\omega} r_j} (1-p)^{\omega(n-\kappa)-\sum_{j=1}^{\omega} (\omega+1-j)r_j}; \tau < X_{\kappa:\kappa:n}. \end{cases} \quad (11)$$

Here, we observe that  $L_1(\delta, \zeta | \underline{x}, \underline{r})$  is independent of  $p$  and  $L_2(p | \underline{r})$  is independent of  $\delta$  and  $\zeta$ . Therefore, we can find the MLEs of  $(\delta, \zeta)$  and  $p$  from Equations (10) and (11) respectively. The log-likelihood functions of Equations (10) and (11) can be written as

$$\log L_1 = \log L_1(\delta, \zeta | \underline{x}, \underline{r}) = \nu \log(\delta) + \nu\delta \log(\zeta) + (\delta-1) \sum_{i=1}^{\nu} \log(x_{i:\kappa:n}) \\ + \sum_{i=1}^{\nu} (\zeta x_{i:\kappa:n})^\delta + \sum_{i=1}^{\nu} (r_i+1) \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) + \nu r_\nu^* \left( 1 - e^{(\zeta \tau^*)^\delta} \right). \quad (12)$$

$$\log L_2 = \log L_2(p | \underline{r}) = \begin{cases} \sum_{j=1}^{\kappa-1} r_j \log(p) + \left[ (\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j)r_j \right] \\ \times \log(1-p); \tau \geq X_{\kappa:\kappa:n}, \\ \sum_{j=1}^{\omega} r_j \log(p) + \left[ \omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega+1-j)r_j \right] \\ \times \log(1-p); \tau < X_{\kappa:\kappa:n}. \end{cases} \quad (13)$$

To determine the ML estimators of  $\delta$ ,  $\zeta$  and  $p$ , we partially differentiate  $\log L_1$  with respect  $\delta$  and  $\zeta$  and  $\log L_2$  with respect to  $p$  and then equalize them to zero. We observe

$$\begin{aligned} \frac{\partial}{\partial \delta} \log L_1 = 0 \Rightarrow & \frac{\nu}{\delta} + \nu \log(\zeta) + \sum_{i=1}^{\nu} \log(x_{i:\kappa:n}) + \sum_{i=1}^{\nu} (\zeta x_{i:\kappa:n})^{\delta} \log(\zeta x_{i:\kappa:n}) \\ & - \sum_{i=1}^{\nu} (r_i + 1) (\zeta x_{i:\kappa:n})^{\delta} e^{(\zeta x_{i:\kappa:n})^{\delta}} \log(\zeta x_{i:\kappa:n}) \\ & - \nu r_{\nu}^* (\zeta \tau^*)^{\delta} e^{(\zeta \tau^*)^{\delta}} \log(\zeta \tau^*) = 0. \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\partial}{\partial \zeta} \log L_1 = 0 \Rightarrow & \frac{\nu \delta}{\zeta} + \frac{\delta}{\zeta} \sum_{i=1}^{\nu} (\zeta x_{i:\kappa:n})^{\delta} - \frac{\delta}{\zeta} \sum_{i=1}^{\nu} (r_i + 1) (\zeta x_{i:\kappa:n})^{\delta} e^{(\zeta x_{i:\kappa:n})^{\delta}} \\ & - \frac{\delta \nu}{\zeta} r_{\nu}^* (\zeta \tau^*)^{\delta} e^{(\zeta \tau^*)^{\delta}} = 0. \end{aligned} \quad (15)$$

$$\frac{\partial}{\partial p} \log L_2 = 0 \Rightarrow \begin{cases} \frac{\sum_{j=1}^{\kappa-1} r_j}{p} - \frac{(\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j) r_j}{(1-p)} = 0; & \tau \geq X_{\kappa:\kappa:n}, \\ \frac{\sum_{j=1}^{\omega} r_j}{p} - \frac{\omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega+1-j) r_j}{(1-p)} = 0; & \tau < X_{\kappa:\kappa:n}. \end{cases} \quad (16)$$

The normal Equations (14) and (15) are non-linear so we cannot find the ML estimators of  $\delta$  and  $\zeta$  in closed form. Therefore, we use numerical techniques such as Newton-Raphson, Nelder-Mead, etc. to solve them. However, the normal Equation (16) is linear and we can find the ML estimator of  $p$  as

$$\hat{p} = \begin{cases} \frac{\sum_{j=1}^{\kappa-1} r_j}{(\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j-1) r_j}; & \tau \geq X_{\kappa:\kappa:n}, \\ \frac{\sum_{j=1}^{\omega} r_j}{\omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega-j) r_j}; & \tau < X_{\kappa:\kappa:n}. \end{cases} \quad (17)$$

To check if there exist a maxima at a point  $(\delta_0, \zeta_0, p_0)$ , we use second derivative test. According to this test, there exist maxima if leading principal minors  $D_1 < 0$ ,  $D_2 > 0$  and  $D_3 < 0$  of hessian matrix  $H$  where  $H$  is defined as

$$H = \begin{bmatrix} \frac{\partial^2 \log L}{\partial \delta^2} & \frac{\partial^2 \log L}{\partial \delta \partial \zeta} & \frac{\partial^2 \log L}{\partial \delta \partial p} \\ \frac{\partial^2 \log L}{\partial \zeta \partial \delta} & \frac{\partial^2 \log L}{\partial \zeta^2} & \frac{\partial^2 \log L}{\partial \zeta \partial p} \\ \frac{\partial^2 \log L}{\partial p \partial \delta} & \frac{\partial^2 \log L}{\partial p \partial \zeta} & \frac{\partial^2 \log L}{\partial p^2} \end{bmatrix},$$

and

$$\begin{aligned} \frac{\partial^2 \log L}{\partial \delta^2} = & -\frac{\nu}{\delta^2} + \sum_{i=1}^{\nu} (\zeta x_{i:\kappa:n})^{\delta} \{\log(\zeta x_{i:\kappa:n})\}^2 - \sum_{i=1}^{\nu} (r_i + 1) e^{(\zeta x_{i:\kappa:n})^{\delta}} (\zeta x_{i:\kappa:n})^{\delta} \\ & \times \left\{ 1 + (\zeta x_{i:\kappa:n})^{\delta} \right\} \{\log(\zeta x_{i:\kappa:n})\}^2 - \nu r_{\nu}^* e^{(\zeta \tau^*)^{\delta}} (\zeta \tau^*)^{\delta} \left\{ 1 + (\zeta \tau^*)^{\delta} \right\} \\ & \times \{\log(\zeta \tau^*)\}^2, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \log L}{\partial \zeta^2} &= -\frac{\nu \delta}{\zeta^2} + \frac{\delta(\delta-1)}{\zeta^2} \sum_{i=1}^{\nu} (\zeta x_{i:\kappa:n})^{\delta} - \frac{\delta}{\zeta^2} \sum_{i=1}^{\nu} \left[ (r_i + 1) e^{(\zeta x_{i:\kappa:n})^{\delta}} (\zeta x_{i:\kappa:n})^{\delta} \right. \\ &\quad \times \left. \left\{ \delta - 1 + \delta(\zeta x_{i:\kappa:n})^{\delta} \right\} \right] - \frac{\delta \nu}{\zeta^2} r_{\nu}^* e^{(\zeta \tau^*)^{\delta}} (\zeta \tau^*)^{\delta} \left\{ \delta - 1 + \delta(\zeta \tau^*)^{\delta} \right\}, \\ \frac{\partial^2 \log L}{\partial p^2} &= \begin{cases} -\frac{\sum_{j=1}^{\kappa-1} r_j}{p^2} - \frac{(\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j)r_j}{(1-p)^2}; & \tau \geq X_{\kappa:\kappa:n}, \\ \frac{\sum_{j=1}^{\omega} r_j}{p^2} - \frac{\omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega+1-j)r_j}{(1-p)^2}; & \tau < X_{\kappa:\kappa:n}, \end{cases} \\ \frac{\partial^2 \log L}{\partial \delta \partial \zeta} &= \frac{\partial^2 \log L}{\partial \zeta \partial \delta} = \frac{\nu}{\zeta} + \frac{1}{\zeta} \sum_{i=1}^{\nu} (\zeta x_{i:\kappa:n})^{\delta} \{1 + \delta \log(\zeta x_{i:\kappa:n})\} \\ &\quad - \frac{1}{\zeta} \sum_{i=1}^{\nu} (r_i + 1) e^{(\zeta x_{i:\kappa:n})^{\delta}} (\zeta x_{i:\kappa:n})^{\delta} \left[ 1 + \delta \log(\zeta x_{i:\kappa:n}) \left\{ 1 + (\zeta x_{i:\kappa:n})^{\delta} \right\} \right] \\ &\quad - \frac{\nu}{\zeta} r_{\nu}^* e^{(\zeta \tau^*)^{\delta}} (\zeta \tau^*)^{\delta} \left[ 1 + \delta \log(\zeta \tau^*) \left\{ 1 + (\zeta \tau^*)^{\delta} \right\} \right], \\ \frac{\partial^2 \log L}{\partial \delta \partial p} &= \frac{\partial^2 \log L}{\partial p \partial \delta} = \frac{\partial^2 \log L}{\partial \zeta \partial p} = \frac{\partial^2 \log L}{\partial p \partial \zeta} = 0. \end{aligned}$$

### 3.1 Fisher Information and Asymptotic Confidence Interval (ACI)

The exact distributions of the suggested ML estimators do not accessible in some known standard forms. Therefore, to obtain CIs for  $\delta$ ,  $\zeta$  and  $p$ , we use the asymptotic distribution theory of ML estimators. Within the regularity criteria, the ML estimator  $(\hat{\delta}, \hat{\zeta}, \hat{p})$  is asymptotically normally distributed with mean  $(\delta, \zeta, p)$  and variance-covariance matrix  $\Lambda$  which is given by

$$\Lambda = \begin{bmatrix} \Lambda_{\delta\delta} & \Lambda_{\delta\zeta} & \Lambda_{\delta p} \\ \Lambda_{\zeta\delta} & \Lambda_{\zeta\zeta} & \Lambda_{\zeta p} \\ \Lambda_{p\delta} & \Lambda_{p\zeta} & \Lambda_{pp} \end{bmatrix},$$

where  $\Lambda_{\delta p} = \Lambda_{p\delta} = \Lambda_{\zeta p} = \Lambda_{p\zeta} = 0$  because  $p$  does not depend on  $\delta$  and  $\zeta$ . We can find  $\Lambda$  with the help of observed Fisher information which is given by

$$I_o(\delta, \zeta, p) = \begin{bmatrix} -\frac{\partial^2 \log L}{\partial \delta^2} & -\frac{\partial^2 \log L}{\partial \delta \partial \zeta} & -\frac{\partial^2 \log L}{\partial \delta \partial p} \\ -\frac{\partial^2 \log L}{\partial \zeta \partial \delta} & -\frac{\partial^2 \log L}{\partial \zeta^2} & -\frac{\partial^2 \log L}{\partial \zeta \partial p} \\ -\frac{\partial^2 \log L}{\partial p \partial \delta} & -\frac{\partial^2 \log L}{\partial p \partial \zeta} & -\frac{\partial^2 \log L}{\partial p^2} \end{bmatrix}. \quad (18)$$

Therefore, the variance-covariance matrix  $\Lambda$  can be obtained using Fisher information as

$$\Lambda = I_o^{-1}(\delta, \zeta, p).$$

Therefore, we get

$$\begin{pmatrix} \hat{\delta} \\ \hat{\zeta} \\ \hat{p} \end{pmatrix} \sim N \left[ \begin{pmatrix} \delta \\ \zeta \\ p \end{pmatrix}, I_o^{-1}(\delta, \zeta, p) \right]. \quad (19)$$

Since  $\Lambda = I_o^{-1}(\delta, \zeta, p)$  contains the unknown parameters  $\delta, \zeta$ , and  $p$  so we use the corresponding ML estimates  $(\hat{\delta}, \hat{\zeta}, \hat{p})$  of the parameters  $(\delta, \zeta, p)$  to obtain the estimate of  $\Lambda$ . Thus, the  $100(1-\psi)\%$  ACIs for  $\delta, \zeta$ , and  $p$  are obtained as

$$[\varphi_L, \varphi_U] = \hat{\varphi} \pm z_{\psi/2} \sqrt{\Lambda_{\hat{\varphi}\hat{\varphi}}}, \quad (20)$$

where  $\varphi$  stands for  $\delta, \zeta$  or  $p$  and  $z_{\psi/2}$  is such that

$$\int_{z_{\psi/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \psi.$$

The coverage probabilities of  $\hat{\delta}, \hat{\zeta}$ , and  $\hat{p}$  can be estimated as in [9]

$$p \left[ \left| \frac{\varphi - \hat{\varphi}}{\sqrt{\Lambda_{\hat{\varphi}\hat{\varphi}}}} \right| \leq z_{\psi/2} \right]. \quad (21)$$

## 4 Bayesian Estimation

Our focus in this segment is to get the Bayes estimators with the Bayesian credible intervals of  $\delta, \zeta$  and  $p$  based on Type-I PHCS with binomial removals. The Bayesian principle suggests that a parameter can be viewed as a random variable rather than a fixed constant as seen in the classical approach. In this approach, the observed sample information is used to enhance prior knowledge about a parameter in terms of the posterior density.

### 4.1 Prior and Posterior Distributions

We let that  $\delta, \zeta$  and  $p$  be independently distributed and  $\delta$  and  $\zeta$  have gamma prior distributions whereas the removal probability  $p$  has prior knowledge in the form of a conjugate beta first kind independent to  $\delta$  and  $\zeta$ . The density functions of these priors are given below

$$g_1(\delta) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} e^{-\beta_1 \delta} \delta^{\alpha_1-1}; \quad \delta > 0, \alpha_1, \beta_1 > 0, \quad (22)$$

$$g_2(\zeta) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} e^{-\beta_2 \zeta} \zeta^{\alpha_2-1}; \quad \zeta > 0, \alpha_2, \beta_2 > 0, \quad (23)$$

$$g_3(p) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1} (1-p)^{\beta-1}; \quad 0 < p < 1, \alpha, \beta > 0, \quad (24)$$

where  $\alpha_1, \beta_1, \alpha_2, \beta_2$  and  $\alpha, \beta$  are called hyperparameters and we choose these in such a manner that they reflect prior knowledge about  $\delta, \zeta$  and  $p$ , respectively.

Since  $\delta, \zeta$  and  $p$  are independent to each other so the joint prior PDF of  $(\delta, \zeta, p)$  can be obtained from Equations (22) to (24) as

$$\begin{aligned} g(\delta, \zeta, p) &= g_1(\delta) g_2(\zeta) g_3(p) \\ &= \frac{1}{B(\alpha, \beta)} \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \delta^{\alpha_1-1} \zeta^{\alpha_2-1} e^{-\beta_1 \delta} e^{-\beta_2 \zeta} p^{\alpha-1} (1-p)^{\beta-1}; \\ &\quad \delta, \zeta > 0, 0 < p < 1. \end{aligned} \quad (25)$$

Using the Bayes theorem, we can obtain the joint posterior PDF of  $(\delta, \zeta, p)$  as

$$\pi^*(\delta, \zeta, p | \underline{x}, \underline{r}) = \frac{L(\delta, \zeta, p | \underline{x}, \underline{r}) g(\delta, \zeta, p)}{\int_0^\infty \int_0^\infty \int_0^1 L(\delta, \zeta, p | \underline{x}, \underline{r}) g(\delta, \zeta, p) dp d\zeta d\delta}. \quad (26)$$

By substituting the values of  $L(\delta, \zeta, p | \underline{x}, \underline{r})$  and  $g(\delta, \zeta, p)$  given in Equations (9) and (22) respectively into Equation (26), we get

$$\begin{aligned} \pi^*(\delta, \zeta, p | \underline{x}, \underline{r}) &= C_1^{-1} \delta^{\kappa+\alpha_1-1} \lambda^{\kappa\delta+\alpha_2-1} \prod_{i=1}^{\kappa} x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^\delta - \beta_1 \delta - \beta_2 \zeta} \\ &\quad \times \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \right\}^{(r_i+1)} p^{\alpha + \sum_{j=1}^{\kappa-1} r_j - 1} \\ &\quad \times (1-p)^{\beta + (\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j)r_j - 1}; \quad \tau \geq X_{\kappa:\kappa:n}, \end{aligned} \quad (27)$$

where

$$C_1 = \int_0^\infty \int_0^\infty \int_0^1 \left[ \delta^{\kappa+\alpha_1-1} \lambda^{\kappa\delta+\alpha_2-1} \prod_{i=1}^{\kappa} x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^\delta - \beta_1 \delta - \beta_2 \zeta} \right. \\ \left. \times \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \right\}^{(r_i+1)} p^{\alpha + \sum_{j=1}^{\kappa-1} r_j - 1} \right. \\ \left. \times (1-p)^{\beta + (\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j)r_j - 1} \right] dp d\lambda d\alpha.$$

$$\begin{aligned} \pi^*(\delta, \zeta, p | \underline{x}, \underline{r}) &= C_2^{-1} \delta^{\omega+\alpha_1-1} \lambda^{\omega\delta+\alpha_2-1} \prod_{i=1}^{\omega} \left[ x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^\delta - \beta_1 \delta - \beta_2 \zeta} \right. \\ &\quad \left. \times \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \right\}^{(r_i+1)} \right] \\ &\quad \times \left\{ \exp \left( 1 - e^{(\zeta \tau)^\delta} \right) \right\}^{r_\omega^*} p^{\alpha + \sum_{j=1}^{\omega} r_j - 1} \\ &\quad \times (1-p)^{\beta + \omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega+1-j)r_j - 1}; \quad \tau < X_{\kappa:\kappa:n}, \end{aligned}$$

where

$$C_2 = \int_0^\infty \int_0^\infty \int_0^1 \left[ \delta^{\omega+\alpha_1-1} \lambda^{\omega\delta+\alpha_2-1} \right. \\ \left. \times \prod_{i=1}^{\omega} \left[ x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^\delta - \beta_1 \delta - \beta_2 \zeta} \right. \right. \\ \left. \left. \times \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \right\}^{(r_i+1)} \right] \right. \\ \left. \times \left\{ \exp \left( 1 - e^{(\zeta \tau)^\delta} \right) \right\}^{r_\omega^*} p^{\alpha + \sum_{j=1}^{\omega} r_j - 1} \right. \\ \left. \times (1-p)^{\beta + \omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega+1-j)r_j - 1} \right] dp d\lambda d\alpha.$$

The  $C_1$  and  $C_2$  cannot be solved analytically due to multi-dimensional parametric spaces, so we cannot obtain the Bayes estimators of  $\delta$ ,  $\zeta$  and  $p$  in explicit form. To overcome this difficulty, we generate sample observations from the posterior distribution using the MCMC methodology and then use these generated observations to compute the Bayes estimates of  $\delta$ ,  $\zeta$  and  $p$  under the squared error loss functions (SELF) [4].

Here, we consider the Gibbs sampling of the MCMC technique given by [5]. In the Gibbs sampling technique, we use a conditional posterior distribution of a parameter to generate random

observations. So first we find the conditional posterior PDF of each parameter. We can write the conditional posterior PDF of  $\delta$  given  $\zeta$ ,  $p$  and data as

$$\pi_1^*(\delta | \zeta, p, \underline{x}, \underline{r}) \propto \begin{cases} \delta^{\kappa+\alpha_1-1} \zeta^{\kappa\delta} \prod_{i=1}^{\kappa} x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^\delta - \beta_1 \delta} \\ \quad \times \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \right\}^{(r_i+1)}; \tau \geq X_{\kappa:\kappa:n}, 0 < \delta < \infty, \\ \delta^{\omega+\alpha_1-1} \zeta^{\omega\delta} \prod_{i=1}^{\omega} x_{i:\kappa:n}^{\delta-1} e^{(\zeta x_{i:\kappa:n})^\delta - \beta_1 \delta} \\ \quad \times \left\{ \exp \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \right\}^{(r_i+1)} \left\{ \exp \left( 1 - e^{(\zeta \tau)^\delta} \right) \right\}^{r_\omega^*}; \\ \quad \tau < X_{\kappa:\kappa:n}, 0 < \delta < \infty. \end{cases} \quad (28)$$

Similarly, the conditional posterior PDF of  $\zeta$  given  $\delta$ ,  $p$  and data can be written as

$$\pi_2^*(\zeta | \delta, p, \underline{x}, \underline{r}) \propto \begin{cases} \zeta^{\kappa\delta+\alpha_2-1} \prod_{i=1}^{\kappa} \left[ e^{(\zeta x_{i:\kappa:n})^\delta - \beta_2 \zeta} \exp \left\{ \begin{array}{c} \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) \\ \times (r_i + 1) \end{array} \right\} \right]; \\ \quad \tau \geq X_{\kappa:\kappa:n}, 0 < \zeta < \infty, \\ \zeta^{\omega\delta+\alpha_2-1} \prod_{i=1}^{\omega} \left[ e^{(\zeta x_{i:\kappa:n})^\delta - \beta_2 \zeta} \right. \\ \quad \times \exp \left\{ \begin{array}{c} \left( 1 - e^{(\zeta x_{i:\kappa:n})^\delta} \right) (r_i + 1) \\ \left. + \left( 1 - e^{(\zeta \tau)^\delta} \right) r_i^* \right\} \right]; \\ \quad \tau < X_{\kappa:\kappa:n}, 0 < \zeta < \infty. \end{cases} \quad (29)$$

Similarly, the conditional posterior PDF of  $p$  given  $\delta$ ,  $\zeta$  and data can be expressed as

$$\pi_3^*(p | \delta, \zeta, \underline{x}, \underline{r}) \propto \begin{cases} p^{\alpha + \sum_{j=1}^{\kappa-1} r_j - 1} (1-p)^{\beta + (\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j)r_j - 1}; \\ \quad \tau \geq X_{\kappa:\kappa:n}, 0 < p < 1, \\ p^{\alpha + \sum_{j=1}^{\omega} r_j - 1} (1-p)^{\beta + \omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega+1-j)r_j - 1}; \\ \quad \tau < X_{\kappa:\kappa:n}, 0 < p < 1. \end{cases} \quad (30)$$

From Equation (30), we observe that  $\pi_3^*(p | \delta, \zeta, \underline{x}, \underline{r})$  follows a Beta distribution of first kind with parameters

$$\begin{cases} \left( \alpha^* = \alpha + \sum_{j=1}^{\kappa-1} r_j, \beta^* = \beta + (\kappa-1)(n-\kappa) - \sum_{j=1}^{\kappa-1} (\kappa-j)r_j \right); \text{ if } \tau \geq X_{\kappa:\kappa:n}, \\ \left( \alpha^{**} = \alpha + \sum_{j=1}^{\omega} r_j, \beta^{**} = \beta + \omega(n-\kappa) - \sum_{j=1}^{\omega} (\omega+1-j)r_j \right); \text{ if } \tau < X_{\kappa:\kappa:n}. \end{cases}$$

Therefore, we can generate observations from  $\pi_3^*(p | \delta, \zeta, \underline{x}, \underline{r})$  using the standard random generation method. However,  $\pi_1^*(\delta | \zeta, p, \underline{x}, \underline{r})$  and  $\pi_2^*(\zeta | \delta, p, \underline{x}, \underline{r})$  are not in standard form so we use the Metropolis-Hastings (M-H) algorithm ([6]) to generate the observations. Since the plots of the conditional posterior PDFs of  $\delta$  and  $\zeta$  are similar to the normal distribution, therefore, we consider normal proposal distribution for both  $\delta$  and  $\zeta$  for the M-H algorithm.

## 4.2 Bayesian Credible Interval (BCI)

The credible interval is a central portion of the posterior distribution that contains a particular percentage of the probable value of a parameter. For computing BCI, we use the algorithm given by [2] which is mentioned in points 8 and 9 of the Gibbs algorithm.

### 4.3 Gibbs Algorithm

To compute the Bayes estimates and BCIs of  $\delta$ ,  $\zeta$ , and  $p$ , we proceed as:

1. Set the initial values  $(\delta^{(0)}, \zeta^{(0)}, p^{(0)})$ .
2. Generate  $\delta^{(1)}$  from Equation (28) using the M-H algorithm by taking  $\zeta = \zeta^{(0)}$ .
3. Generate  $\zeta^{(1)}$  from Equation (29) using the M-H algorithm by taking  $\delta = \delta^{(1)}$ .
4. Generate  $p^{(1)}$  from Equation (30) using a standard random generation method from Beta first kind.
5. Repeat steps from 2 to 4,  $N$  times and note the sequence of  $(\delta, \zeta, p)$ .
6. To eliminate the impact of starting values, take burn-in of first  $K$  iterations and get the sequence of the parameters  $(\delta^{(i)}, \zeta^{(i)}, p^{(i)})$ ;  $i = K + 1, K + 2, \dots, N$ .
7. The Bayes estimates  $\delta^*$ ,  $\zeta^*$  and  $p^*$  of  $\delta$ ,  $\zeta$  and  $p$  under SELF are obtained as

$$\varphi^* = \frac{1}{N - K} \sum_{i=K+1}^N \varphi^{(i)},$$

where  $\varphi$  stands for  $\delta$ ,  $\zeta$  or  $p$ .

8. For Bayesian credible interval, consider the ordered observations as  $\phi_{(K+1)} \leq \phi_{(K+2)} \leq \dots \leq \phi_{(N)}$ , where  $\phi_{(K+1)}, \phi_{(K+2)}, \dots, \phi_{(N)}$  denote the ordered observations of  $\phi^{(K+1)}, \phi^{(K+2)}, \dots, \phi^{(N)}$ .
9. Calculate  $100(1 - \psi)\%$  credible interval for the parameters  $\delta$ ,  $\zeta$  and  $p$  as

$$\left( \phi_{[(\frac{\psi}{2})(N-K)]}, \phi_{[(1-\frac{\psi}{2})(N-K)]} \right),$$

where  $[t]$  indicates an integral part of  $t$ .

## 5 Simulation Study

In this segment, we performed a simulation study to evaluate the effectiveness of the estimators because the ML and Bayes estimators are not in closed form. Without loss of generality, a simulation study is performed for  $\delta=2$  and  $\zeta=0.5$  and different removals probabilities;  $p=0.25, 0.50, 0.75$ , sample size;  $n=30$ , different effective sample sizes,  $\kappa=9, 18, 27$  and different termination times;  $\tau=1.5, 3$ . We choose  $\kappa$  in such a way that it yields 70%, 50% and 10% censored data when the time restrictions are lacking. Also, we choose the termination time  $\tau$  that it ensures nearly 70% and 10% censored data when the failure restrictions are lacking. The performance of the proposed estimators is compared according to their absolute biases and MSEs based on Type-I PH censored data. Also, we obtain the confidence intervals with coverage probability and BCIs for the parameters.

We use the Monte Carlo simulation technique to get the ML estimates of  $\delta$ ,  $\zeta$  and  $p$  and generate  $N=5,000$  samples for each combination under Type-I PH censored data. Then we find ML estimate, absolute bias (AB) and MSE for each generated sample and then take the average of these

corresponding ML estimates as well as MSEs to find the final ML estimate and MSE, respectively of the parameters i.e.

$$\hat{\varphi} = \frac{1}{N} \sum_{i=1}^N \hat{\varphi}_i, \quad AB(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^N |\hat{\varphi}_i - \varphi|$$

and

$$MSE(\hat{\varphi}) = \frac{1}{N} \sum_{i=1}^N (\hat{\varphi}_i - \varphi)^2,$$

where  $\varphi$  stands for  $\delta$ ,  $\zeta$  or  $p$ .

We calculate AB of ML estimates along with their MSEs and given in Table 1. Also, we find the average width of 95% confidence interval (CI) with coverage probabilities (CP) for  $\delta$ ,  $\zeta$  and  $p$  in the same table. For test their exist maxima, we calculate leading principle minors  $D_1$ ,  $D_2$  and  $D_3$  for each combination in Table 1.

For simulation study of the Bayes estimators, we select hyperparameters of the prior distributions of  $\delta$ ,  $\zeta$  and  $p$  in a manner that the mean of prior distributions equal to the true value of the related parameters i.e. we take  $\delta = \alpha_1/\beta_1$ ,  $\zeta = \alpha_2/\beta_2$  and  $p = \alpha/(\alpha + \beta)$  and they produce informative priors. We repeat the Gibbs algorithm  $N = 10,000$  iterations with 2,000 iterations as burn-in. We calculate Bayes estimates of  $\delta$ ,  $\zeta$  and  $p$  with their corresponding risk for each parameters combination and given in Table 2. All the computational work is done in R studio 1.3.959 ([14]).

Table 1: Comparative study of the performance of ML estimators of the parameters with their ABs and MSEs and average CI widths.

| $\tau$ | $p$  | $\kappa$ | Shape parameter ( $\delta$ ) |        |            |        | Scale parameter ( $\zeta$ ) |        |            |        | Binomial parameter ( $p$ ) |        |            |        | $D_1$  | $D_2(10^4)$ | $D_3(10^7)$ |
|--------|------|----------|------------------------------|--------|------------|--------|-----------------------------|--------|------------|--------|----------------------------|--------|------------|--------|--------|-------------|-------------|
|        |      |          | AB                           | MSE    | CI (width) | CP     | AB                          | MSE    | CI (width) | CP     | AB                         | MSE    | CI (width) | CP     |        |             |             |
| 1.5    | 0.25 | 9        | 0.2634                       | 0.6454 | 2.2843     | 0.9230 | 0.5217                      | 0.0217 | 0.0131     | 0.2929 | 0.7804                     | 0.0077 | 0.0028     | 0.2007 | -5.552 | 0.188       | -0.763      |
|        |      | 18       | 0.1567                       | 0.3900 | 2.0292     | 0.9354 | 0.5038                      | 0.0038 | 0.0064     | 0.2388 | 0.8400                     | 0.0139 | 0.0052     | 0.2581 | -6.349 | 0.283       | -0.728      |
|        |      | 27       | 0.1427                       | 0.3696 | 1.9168     | 0.9362 | 0.5029                      | 0.0029 | 0.0044     | 0.1986 | 0.8392                     | 0.0700 | 0.0345     | 0.5466 | -6.916 | 0.406       | -2.433      |
|        | 0.5  | 9        | 0.1954                       | 0.5572 | 2.3486     | 0.9302 | 0.5119                      | 0.0119 | 0.0129     | 0.3307 | 0.8136                     | 0.0138 | 0.0068     | 0.3037 | -5.184 | 0.140       | -0.238      |
|        |      | 18       | 0.1348                       | 0.4117 | 2.0772     | 0.9290 | 0.5008                      | 0.0008 | 0.0065     | 0.2472 | 0.8500                     | 0.0206 | 0.0116     | 0.3957 | -5.992 | 0.254       | -0.256      |
|        |      | 27       | 0.1344                       | 0.3569 | 1.9176     | 0.9276 | 0.4999                      | 0.0001 | 0.0045     | 0.2017 | 0.8490                     | 0.0717 | 0.0510     | 0.7588 | -6.856 | 0.401       | -0.076      |
| 0.75   | 0.25 | 9        | 0.1640                       | 0.5654 | 2.4231     | 0.9322 | 0.5023                      | 0.0023 | 0.0135     | 0.3465 | 0.8264                     | 0.0095 | 0.0066     | 0.3083 | -4.865 | 0.121       | -0.202      |
|        |      | 18       | 0.1461                       | 0.4443 | 2.1277     | 0.9322 | 0.5007                      | 0.0007 | 0.0070     | 0.2466 | 0.8386                     | 0.0149 | 0.0112     | 0.4015 | -5.743 | 0.244       | -0.187      |
|        |      | 27       | 0.1447                       | 0.3660 | 1.9328     | 0.9330 | 0.5007                      | 0.0007 | 0.0045     | 0.2004 | 0.8404                     | 0.0520 | 0.0390     | 0.8367 | -6.842 | 0.395       | -0.040      |
|        | 0.5  | 9        | 0.3544                       | 0.6503 | 2.1884     | 0.9236 | 0.5431                      | 0.0431 | 0.0103     | 0.2221 | 0.7690                     | 0.0061 | 0.0028     | 0.1972 | -4.899 | 0.247       | -1.025      |
|        |      | 18       | 0.1589                       | 0.2004 | 1.5329     | 0.9544 | 0.5124                      | 0.0124 | 0.0018     | 0.1247 | 0.8592                     | 0.0163 | 0.0053     | 0.2551 | -7.410 | 0.906       | -2.420      |
|        |      | 27       | 0.1128                       | 0.1351 | 1.3225     | 0.9546 | 0.5062                      | 0.0062 | 0.0010     | 0.0999 | 0.8838                     | 0.0746 | 0.0361     | 0.5486 | -9.778 | 1.790       | -1.370      |
| 3.0    | 0.5  | 9        | 0.1527                       | 0.3696 | 1.9168     | 0.9352 | 0.5029                      | 0.0029 | 0.0044     | 0.1986 | 0.8392                     | 0.0700 | 0.0345     | 0.5466 | -4.861 | 0.308       | -0.527      |
|        |      | 18       | 0.1540                       | 0.1953 | 1.5417     | 0.9534 | 0.5114                      | 0.0114 | 0.0018     | 0.1224 | 0.8564                     | 0.0205 | 0.0112     | 0.3958 | -7.321 | 0.940       | -0.946      |
|        |      | 27       | 0.1223                       | 0.1474 | 1.3333     | 0.9524 | 0.5067                      | 0.0067 | 0.0010     | 0.0996 | 0.8756                     | 0.0770 | 0.0529     | 0.7596 | -9.655 | 1.798       | -0.426      |
|        | 0.75 | 9        | 0.2398                       | 0.3778 | 1.9991     | 0.9546 | 0.5268                      | 0.0268 | 0.0049     | 0.1773 | 0.8168                     | 0.0097 | 0.0066     | 0.3080 | -4.658 | 0.323       | -0.541      |
|        |      | 18       | 0.1452                       | 0.1979 | 1.5585     | 0.9506 | 0.5109                      | 0.0109 | 0.0017     | 0.1227 | 0.8638                     | 0.0162 | 0.0113     | 0.4009 | -7.136 | 0.935       | -0.901      |
|        |      | 27       | 0.1269                       | 0.1498 | 1.3397     | 0.9482 | 0.5072                      | 0.0072 | 0.0010     | 0.0994 | 0.8716                     | 0.0508 | 0.0378     | 0.8371 | -9.657 | 1.786       | -0.223      |

From Table 1, we observe that  $D_1 < 0$ ,  $D_2 > 0$  and  $D_3 < 0$  for all combinations so we conclude that there exist maxima at each combination.

Table 2: Comparative study of the performance of Bayes estimators of the parameters with their ABs and MSEs and average BCI widths.

| $\tau$ | $p$  | $\kappa$ | Shape parameter ( $\delta$ ) |        |             | Scale parameter ( $\zeta$ ) |        |             | Binomial parameter ( $p$ ) |        |             |
|--------|------|----------|------------------------------|--------|-------------|-----------------------------|--------|-------------|----------------------------|--------|-------------|
|        |      |          | AB                           | MSE    | BCI (width) | AB                          | MSE    | BCI (width) | AB                         | MSE    | BCI (width) |
| 1.5    | 0.25 | 9        | 0.0214                       | 0.2044 | 1.4825      | -0.0252                     | 0.0107 | 0.1820      | 0.0049                     | 0.0026 | 0.1959      |
|        |      | 18       | -0.0152                      | 0.1659 | 1.3417      | -0.0225                     | 0.0057 | 0.1395      | 0.0144                     | 0.0055 | 0.2515      |
|        |      | 27       | -0.0105                      | 0.1624 | 1.2870      | -0.0206                     | 0.0044 | 0.1177      | 0.0474                     | 0.0208 | 0.4759      |
|        | 0.50 | 9        | -0.0309                      | 0.1927 | 1.4583      | -0.0350                     | 0.0112 | 0.1866      | 0.0106                     | 0.0058 | 0.2928      |
|        |      | 18       | -0.0184                      | 0.1856 | 1.3717      | -0.0295                     | 0.0070 | 0.1403      | 0.0117                     | 0.0098 | 0.3716      |
|        |      | 27       | -0.0107                      | 0.1837 | 1.2862      | -0.0183                     | 0.0046 | 0.1158      | 0.0371                     | 0.0257 | 0.6154      |
|        | 0.75 | 9        | -0.0645                      | 0.1906 | 1.4597      | -0.0441                     | 0.0125 | 0.1836      | 0.0094                     | 0.0063 | 0.2951      |
|        |      | 18       | -0.0064                      | 0.1841 | 1.3935      | -0.0256                     | 0.0065 | 0.1368      | 0.0134                     | 0.0097 | 0.3667      |
|        |      | 27       | -0.0041                      | 0.1600 | 1.2628      | -0.0234                     | 0.0052 | 0.1171      | 0.0347                     | 0.0268 | 0.5412      |
| 3.0    | 0.25 | 9        | 0.1049                       | 0.2220 | 1.4007      | 0.0040                      | 0.0066 | 0.1309      | 0.0037                     | 0.0027 | 0.1925      |
|        |      | 18       | 0.0709                       | 0.1247 | 0.9931      | 0.0037                      | 0.0017 | 0.0438      | 0.0188                     | 0.0056 | 0.2498      |
|        |      | 27       | 0.0391                       | 0.0934 | 0.8675      | 0.0034                      | 0.0010 | 0.0377      | 0.0673                     | 0.0282 | 0.4885      |
|        | 0.50 | 9        | 0.0727                       | 0.1742 | 1.2044      | 0.0057                      | 0.0040 | 0.0759      | 0.0090                     | 0.0056 | 0.2908      |
|        |      | 18       | 0.0436                       | 0.1192 | 0.9856      | 0.0022                      | 0.0014 | 0.0423      | 0.0189                     | 0.0098 | 0.3733      |
|        |      | 27       | 0.0226                       | 0.0896 | 0.8608      | 0.0021                      | 0.0009 | 0.0379      | 0.0370                     | 0.0253 | 0.6168      |
|        | 0.75 | 9        | 0.0641                       | 0.1772 | 1.2113      | 0.0052                      | 0.0039 | 0.0639      | 0.0070                     | 0.0058 | 0.2955      |
|        |      | 18       | 0.0614                       | 0.1292 | 1.0185      | 0.0043                      | 0.0016 | 0.0420      | 0.0150                     | 0.0108 | 0.3636      |
|        |      | 27       | 0.0526                       | 0.1044 | 0.8877      | 0.0036                      | 0.0010 | 0.0376      | 0.0377                     | 0.0263 | 0.5391      |

6 Real-life Data Analysis

In this portion, a real-life dataset is considered to illustrate the applicability of the suggested approach to the real-life phenomena. The dataset is taken from [10]. The dataset represents the time (in weeks) to tumour appearance in the rats.

Table 3: The dataset represents the time (in weeks) to tumour appearance in the rats.

|    |    |    |     |    |     |    |     |    |     |     |    |    |     |    |
|----|----|----|-----|----|-----|----|-----|----|-----|-----|----|----|-----|----|
| 49 | 88 | 96 | 104 | 77 | 96  | 70 | 89  | 39 | 50  | 103 | 67 | 68 | 40  | 81 |
| 64 | 86 | 55 | 34  | 54 | 103 | 73 | 84  | 71 | 102 | 80  | 45 | 94 | 101 | 84 |
| 78 | 80 | 81 | 72  | 73 | 66  | 92 | 102 | 89 | 103 | 75  | 79 |    |     |    |

We checked the validity of exponential power distribution with five other well-known lifetime models: 1. The Exponentiated Weibull, 2. Exponentiated Exponential, 3. Generalized Rayleigh, 4. Generalized Power Weibull and 5. Generalized inverted exponential. To test which model suits the dataset best, we use goodness-of-fit criterion: the log-likelihood function, Akaike’s information criterion (AIC), Bayesian information criterion (BIC) as well as goodness-of-fit statistics: Kolmogorov-Smirnov (K-S), Cramer-Von Mises (C-V M) and Anderson-Darling (A-D) statistics. We obtained the values of all using the dataset in Table 4.

Table 4: ML estimates of the parameters and values of goodness-of-fit criteria and statistics of various fitted models based on time to tumour appearance dataset.

| Fitted Distribution | ML Estimates |        | -Log-Likelihood | AIC     | BIC     | K-S   | C-V M | A-D   |
|---------------------|--------------|--------|-----------------|---------|---------|-------|-------|-------|
|                     | Shape        | Scale  |                 |         |         |       |       |       |
| EPD                 | 3.77         | 0.01   | 180.84*         | 365.69* | 369.16* | 0.09* | 0.04* | 0.39* |
| 1                   | 1031.10      | 0.46   | 192.16          | 388.32  | 391.79  | 0.17  | 0.35  | 2.03  |
| 2                   | 26.45        | 0.05   | 188.93          | 381.87  | 385.34  | 0.15  | 0.24  | 1.48  |
| 3                   | 4.35         | 0.02   | 185.60          | 375.21  | 378.68  | 0.11  | 0.14  | 0.94  |
| 4                   | 3.21         | 0.06   | 269.66          | 543.32  | 546.79  | 0.61  | 4.57  | 20.72 |
| 5                   | 66.19        | 345.30 | 185.71          | 375.41  | 378.89  | 0.10  | 0.14  | 1.06  |

\* indicates the minimum value.

From Table 4, we observe that the values of all measures corresponding to the EPD in comparison to other fitted models are minimised which indicate that the EPD is the best fit for the considered dataset among all considered well-known distribution.

6.1 Estimation of the Parameters Under Type-I PHCS with Binomial Removals Based on the Real-life Dataset

We generate samples from the considered dataset under the proposed method for various combinations of the removals probability;  $p=0.25, 0.50, 0.75$ , effective sample size;  $\kappa=38\%, 57\%, 76\%$  of the sample sizes, sampling schemes and termination time  $\tau=75, 105$  and shown in Table 5.

Table 5: Censored samples taken from the tumour appearance rat dataset under Type-I PHCS with binomial removals.

|                                       |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
|---------------------------------------|----|-----|-----|-----|-----|-----|-----|----|----|-----|-----|-----|-----|-----|-----|
| $(\kappa = 16, p = 0.25, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 50  | 55  | 64  | 66  | 68  | 70 | 72 |     |     |     |     |     |     |
| $(\kappa = 24, p = 0.25, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 45  | 54  | 55  | 64  | 67  | 68 | 71 | 73  | 73  |     |     |     |     |
| $(\kappa = 32, p = 0.25, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 45  | 50  | 54  | 64  | 67  | 68 | 70 | 71  | 72  | 73  | 73  |     |     |
| $(\kappa = 16, p = 0.50, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 45  | 49  | 50  | 71  | 73  |    |    |     |     |     |     |     |     |
| $(\kappa = 24, p = 0.50, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 45  | 49  | 54  | 55  | 64  | 66 | 70 | 71  | 73  |     |     |     |     |
| $(\kappa = 32, p = 0.50, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 49  | 50  | 64  | 66  | 68  | 70 | 72 | 73  |     |     |     |     |     |
| $(\kappa = 16, p = 0.75, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 45 | 50  | 67  | 70  |     |     |     |    |    |     |     |     |     |     |     |
| $(\kappa = 24, p = 0.75, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 45  | 49  | 64  | 66  | 67  | 68  | 71 | 72 | 73  |     |     |     |     |     |
| $(\kappa = 32, p = 0.75, \tau = 75)$  |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 49  | 50  | 54  | 55  | 64  | 66 | 67 | 68  | 71  | 73  | 73  |     |     |
| $(\kappa = 16, p = 0.25, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 50  | 66  | 67  | 71  | 73  | 73 | 77 | 80  | 80  | 81  | 84  | 89  | 102 |
| $(\kappa = 24, p = 0.25, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 50  | 54  | 64  | 67  | 68  | 70 | 71 | 72  | 73  | 75  | 77  | 78  | 80  |
| 84                                    | 88 | 89  | 89  | 94  | 96  | 102 | 102 |    |    |     |     |     |     |     |     |
| $(\kappa = 32, p = 0.25, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 49  | 54  | 55  | 64  | 66  | 67 | 68 | 70  | 71  | 72  | 73  | 73  | 77  |
| 79                                    | 80 | 80  | 81  | 84  | 84  | 86  | 88  | 89 | 89 | 94  | 96  | 96  | 101 | 102 | 102 |
| $(\kappa = 16, p = 0.50, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 49  | 64  | 72  | 73  | 73  | 80 | 81 | 88  | 96  | 102 | 102 | 103 | 104 |
| $(\kappa = 24, p = 0.50, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 45 | 49  | 54  | 55  | 64  | 66  | 67  | 70 | 73 | 78  | 79  | 80  | 81  | 84  | 86  |
| 89                                    | 89 | 92  | 94  | 96  | 101 | 102 | 104 |    |    |     |     |     |     |     |     |
| $(\kappa = 32, p = 0.50, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 45  | 49  | 55  | 64  | 66  | 67 | 68 | 70  | 71  | 73  | 73  | 75  | 77  |
| 79                                    | 80 | 80  | 81  | 81  | 88  | 89  | 89  | 92 | 94 | 96  | 96  | 101 | 102 | 102 | 104 |
| $(\kappa = 16, p = 0.75, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 50  | 54  | 55  | 67  | 72  | 75  | 80 | 80 | 86  | 89  | 92  | 96  | 102 | 103 |
| $(\kappa = 24, p = 0.75, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 50  | 72  | 73  | 73  | 75  | 80 | 81 | 81  | 84  | 84  | 89  | 89  | 92  |
| 94                                    | 96 | 101 | 102 | 103 | 103 | 103 | 104 |    |    |     |     |     |     |     |     |
| $(\kappa = 32, p = 0.75, \tau = 105)$ |    |     |     |     |     |     |     |    |    |     |     |     |     |     |     |
| 34                                    | 39 | 40  | 45  | 49  | 50  | 54  | 55  | 66 | 67 | 70  | 71  | 72  | 73  | 73  | 78  |
| 79                                    | 80 | 80  | 81  | 81  | 84  | 89  | 92  | 94 | 96 | 102 | 102 | 103 | 103 | 103 | 104 |

Table 6: ML and interval estimates of the parameters based on the tumour appearance rat dataset for different choices of  $p, \kappa$  and  $\tau$ .

| $\tau$ | $p$  | $\kappa$ | Shape parameter ( $\delta$ ) |        |            | Scale parameter ( $\zeta$ ) |        |            | Binomial parameter ( $p$ ) |        |            | $D_1$  | $D_2(10^7)$ | $D_3(10^{10})$ |
|--------|------|----------|------------------------------|--------|------------|-----------------------------|--------|------------|----------------------------|--------|------------|--------|-------------|----------------|
|        |      |          | ML estimate                  | SE     | CI (width) | ML estimate                 | SE     | CI (width) | ML estimate                | SE     | CI (width) |        |             |                |
| 75     | 0.25 | 16       | 3.7124                       | 0.7714 | 3.0238     | 0.0105                      | 0.0006 | 0.0022     | 0.2539                     | 0.0445 | 0.1745     | -2.218 | 0.802       | -0.431         |
|        |      | 24       | 3.6681                       | 0.7328 | 2.8723     | 0.0105                      | 0.0005 | 0.0020     | 0.2568                     | 0.0528 | 0.2069     | -2.396 | 1.034       | -0.385         |
|        |      | 32       | 3.6609                       | 0.7024 | 2.7534     | 0.0105                      | 0.0005 | 0.0018     | 0.2708                     | 0.0725 | 0.2843     | -2.594 | 1.330       | -0.298         |
|        | 0.50 | 16       | 3.7789                       | 0.8270 | 3.2417     | 0.0106                      | 0.0006 | 0.0024     | 0.5139                     | 0.0696 | 0.2729     | -1.909 | 0.644       | -0.137         |
|        |      | 24       | 3.7238                       | 0.7677 | 3.0094     | 0.0105                      | 0.0005 | 0.0020     | 0.5123                     | 0.0828 | 0.3247     | -2.198 | 0.940       | -0.139         |
|        |      | 32       | 3.6890                       | 0.7188 | 2.8176     | 0.0105                      | 0.0005 | 0.0018     | 0.5250                     | 0.1105 | 0.4332     | -2.474 | 1.271       | -0.109         |
|        | 0.75 | 16       | 3.8266                       | 0.8681 | 3.4031     | 0.0106                      | 0.0006 | 0.0024     | 0.7585                     | 0.0710 | 0.2784     | -1.761 | 0.588       | -0.118         |
|        |      | 24       | 3.7659                       | 0.7902 | 3.0975     | 0.0105                      | 0.0005 | 0.0020     | 0.7571                     | 0.0846 | 0.3316     | -2.089 | 0.896       | -0.132         |
|        |      | 32       | 3.6911                       | 0.7250 | 2.8418     | 0.0105                      | 0.0005 | 0.0018     | 0.7706                     | 0.1109 | 0.4348     | -2.391 | 1.266       | -0.064         |
| 105    | 0.25 | 16       | 3.6093                       | 0.6317 | 2.4761     | 0.0102                      | 0.0003 | 0.0013     | 0.2587                     | 0.0437 | 0.1711     | -2.443 | 2.547       | -1.442         |
|        |      | 24       | 3.7404                       | 0.5842 | 2.2899     | 0.0106                      | 0.0002 | 0.0010     | 0.2610                     | 0.0524 | 0.2056     | -2.964 | 5.117       | -2.016         |
|        |      | 32       | 3.7560                       | 0.5438 | 2.1317     | 0.0106                      | 0.0002 | 0.0008     | 0.2676                     | 0.0711 | 0.2787     | -3.419 | 8.110       | -1.824         |
|        | 0.50 | 16       | 3.8386                       | 0.6774 | 2.6552     | 0.0106                      | 0.0003 | 0.0012     | 0.5098                     | 0.0691 | 0.2708     | -2.327 | 2.790       | -0.593         |
|        |      | 24       | 3.7901                       | 0.6027 | 2.3627     | 0.0106                      | 0.0002 | 0.0009     | 0.5151                     | 0.0830 | 0.3252     | -2.821 | 5.166       | -0.766         |
|        |      | 32       | 3.7753                       | 0.5514 | 2.1615     | 0.0106                      | 0.0002 | 0.0008     | 0.5183                     | 0.1099 | 0.4309     | -3.360 | 8.077       | -0.682         |
|        | 0.75 | 16       | 3.8583                       | 0.6964 | 2.7300     | 0.0106                      | 0.0003 | 0.0011     | 0.7558                     | 0.0713 | 0.2797     | -2.161 | 2.773       | -0.562         |
|        |      | 24       | 3.8115                       | 0.6150 | 2.4107     | 0.0106                      | 0.0002 | 0.0009     | 0.7590                     | 0.0844 | 0.3309     | -2.719 | 5.155       | -0.747         |
|        |      | 32       | 3.8054                       | 0.5594 | 2.1928     | 0.0106                      | 0.0002 | 0.0008     | 0.7688                     | 0.1121 | 0.4394     | -3.292 | 8.060       | -0.065         |

Table 7: Bayes and BCI estimates of the parameters based on the tumour appearance dataset for different choices of  $p$ ,  $\kappa$ , and  $\tau$ .

| $\tau$ | $p$  | $\kappa$ | Shape parameter ( $\delta$ ) |        |             | Scale parameter ( $\zeta$ ) |        |             | Binomial parameter ( $p$ ) |        |             |
|--------|------|----------|------------------------------|--------|-------------|-----------------------------|--------|-------------|----------------------------|--------|-------------|
|        |      |          | Bayes estimate               | SE     | BCI (width) | Bayes estimate              | SE     | BCI (width) | Bayes estimate             | SE     | BCI (width) |
| 75     | 0.25 | 16       | 3.6544                       | 0.0358 | 0.1702      | 0.0103                      | 0.0008 | 0.0033      | 0.2563                     | 0.0446 | 0.1742      |
|        |      | 24       | 3.5522                       | 0.0337 | 0.1675      | 0.0102                      | 0.0008 | 0.0029      | 0.2607                     | 0.0529 | 0.2062      |
|        |      | 32       | 3.6989                       | 0.0291 | 0.1460      | 0.0103                      | 0.0007 | 0.0026      | 0.2657                     | 0.0705 | 0.2739      |
|        | 0.5  | 16       | 3.6587                       | 0.0339 | 0.1685      | 0.0101                      | 0.0009 | 0.0036      | 0.5063                     | 0.0685 | 0.2672      |
|        |      | 24       | 3.5724                       | 0.0332 | 0.1593      | 0.0101                      | 0.0008 | 0.0031      | 0.5148                     | 0.0817 | 0.3182      |
|        |      | 32       | 3.5848                       | 0.0323 | 0.1568      | 0.0102                      | 0.0007 | 0.0027      | 0.5294                     | 0.1077 | 0.4160      |
|        | 0.75 | 16       | 3.6294                       | 0.0334 | 0.1658      | 0.0101                      | 0.0009 | 0.0037      | 0.7572                     | 0.0704 | 0.2729      |
|        |      | 24       | 3.7522                       | 0.0346 | 0.1723      | 0.0104                      | 0.0008 | 0.0030      | 0.7588                     | 0.0824 | 0.3179      |
|        |      | 32       | 3.6526                       | 0.0296 | 0.1633      | 0.0103                      | 0.0007 | 0.0026      | 0.7671                     | 0.1012 | 0.3853      |
|        | 0.25 | 16       | 3.5747                       | 0.0289 | 0.1423      | 0.0102                      | 0.0005 | 0.0020      | 0.2578                     | 0.0433 | 0.1693      |
|        |      | 24       | 3.6760                       | 0.0241 | 0.1174      | 0.0105                      | 0.0004 | 0.0015      | 0.2590                     | 0.0518 | 0.2019      |
|        |      | 32       | 3.6928                       | 0.0233 | 0.1122      | 0.0105                      | 0.0003 | 0.0013      | 0.2635                     | 0.0700 | 0.2720      |
| 105    | 0.5  | 16       | 3.6973                       | 0.0271 | 0.1384      | 0.0105                      | 0.0005 | 0.0019      | 0.2657                     | 0.0705 | 0.2739      |
|        |      | 24       | 3.7183                       | 0.0237 | 0.1204      | 0.0105                      | 0.0004 | 0.0015      | 0.5113                     | 0.0815 | 0.3170      |
|        |      | 32       | 3.6933                       | 0.0213 | 0.1065      | 0.0105                      | 0.0003 | 0.0013      | 0.5260                     | 0.1071 | 0.4140      |
|        | 0.75 | 16       | 3.8685                       | 0.0271 | 0.1357      | 0.0105                      | 0.0004 | 0.0018      | 0.7576                     | 0.0702 | 0.2722      |
|        |      | 24       | 3.7696                       | 0.0236 | 0.1103      | 0.0105                      | 0.0004 | 0.0014      | 0.7626                     | 0.0817 | 0.3151      |
|        |      | 32       | 3.6341                       | 0.0240 | 0.1125      | 0.0105                      | 0.0003 | 0.0013      | 0.7645                     | 0.1025 | 0.3904      |

Based on the censored samples obtained from the dataset, given in Table 5, we obtained ML and Bayes estimates of  $\delta$ ,  $\zeta$  and  $p$  which are given in Table 6 and 7 respectively. For Bayesian estimation, we considered non-informative priors.

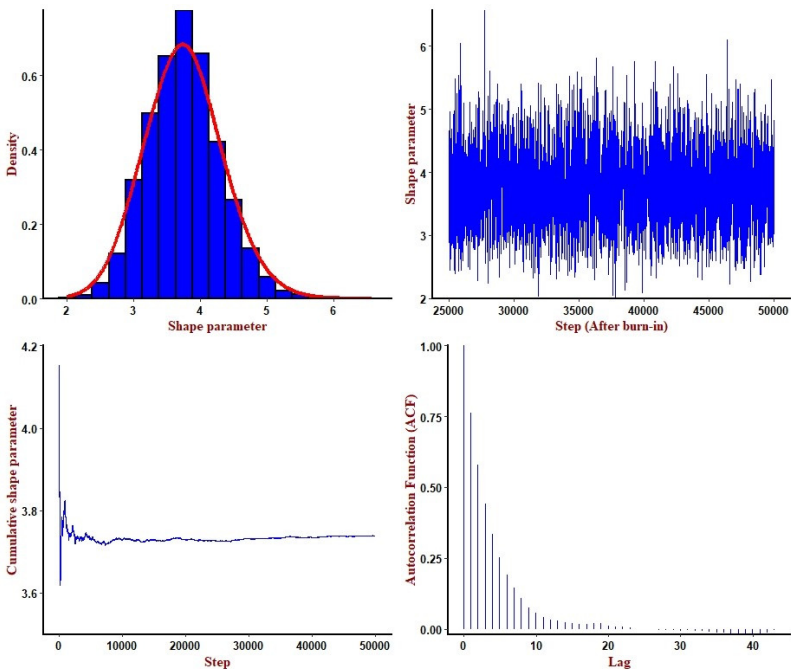


Figure 1: Top left: histogram and posterior density plot; Top right: time series plot (b) Bottom left: cumulative plot; Bottom right: ACF plot of the shape parameter ( $\delta$ ) based on the sampled value after burn-in of the tumour appearance rat dataset.

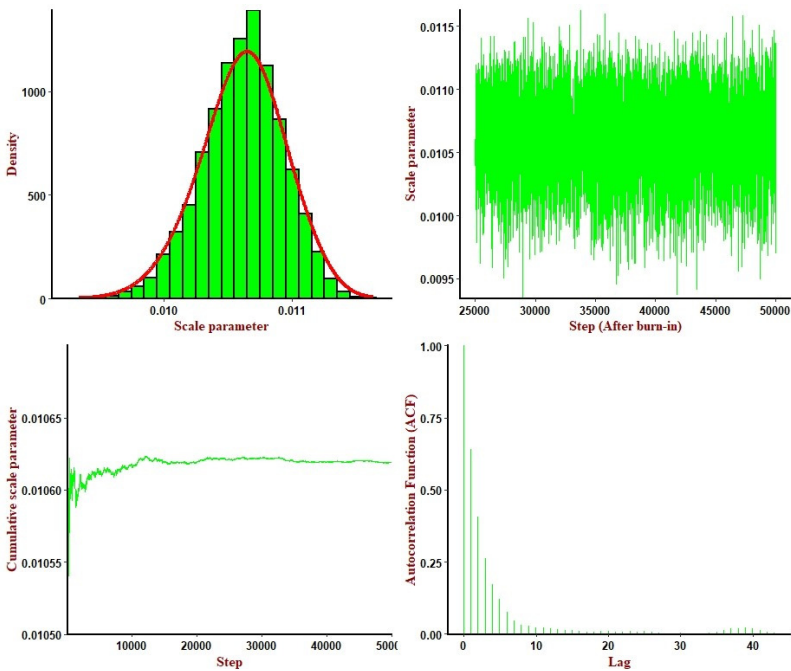


Figure 2: Top left: histogram and posterior density plot; Top right: time series plot (b) Bottom left: cumulative plot; Bottom right: ACF plot of the scale parameter ( $\zeta$ ) based on the sampled value after burn-in of the tumour appearance rat dataset.

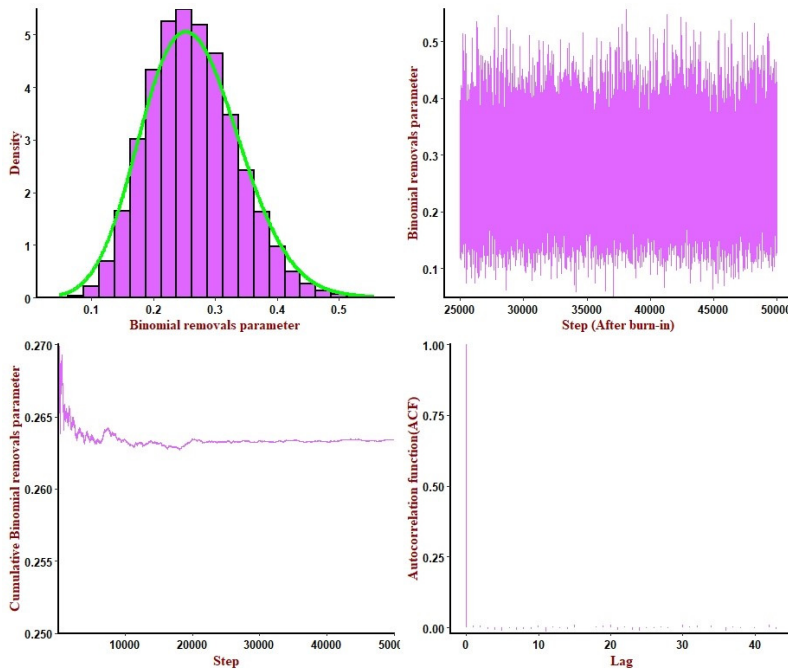


Figure 3: Top left: histogram and posterior density plot; Top right: time series plot (b) Bottom left: cumulative plot; Bottom right: ACF plot (bottom right) of the Binomial removal parameter ( $p$ ) based on the sampled value after burn-in of the tumour appearance rat dataset.

From Figures 1-3, we observed that

- The histogram plots approximate the posterior distribution of the corresponding parameters and the posterior distributions of parameters  $\delta$  and  $p$  are slightly right-skewed whereas for the parameter  $\zeta$  it is left-skewed.
- The plots of time series exhibit good mixing of the Gibbs sampler.
- The cumulative plots show steady convergence to the Bayesian point estimate.
- The ACF plots display the decreasing autocorrelations with lags over 20.

## 7 Conclusions

In this paper, we discussed the estimation in exponential power distribution under the Type-I progressive hybrid censoring with binomial removals. We carried out the classical and Bayesian approaches to estimate the unknown parameters with applications of the present work in real-life. The effectiveness of both approaches is evaluated in terms of their ABs and MSEs. Based on the analysis, we draw the following conclusions:

1. As we increase  $\kappa$  by keeping  $n$ ,  $p$  and  $\tau$  fixed, we observe that the ABs and MSEs of the ML and Bayes estimators decrease for all combinations. Also, we observed the same phenomenon in the comparative study of the average widths of confidence intervals and the BCIs while the coverage probabilities converge to the nominal value. This can be happened

due to the fact in this censoring scheme as  $\kappa$  increases some additional information is gathered. This suggests that effective sample size has a major role to play in determining the efficiency of the estimators.

2. As we increase  $\tau$  by keeping  $n, p$  and  $\kappa$  fixed then we observe the similar results as in case of increasing  $\kappa$ . It is attributed as censorship time raises the amount observed failures increases too. This indicates that the performance of the estimators improves as the termination time increases.
3. As we increase  $p$  by keeping  $n, \tau$  and  $\kappa$  fixed, we get a reverse trend, that is, ABs and MSEs of the estimators increase. Similarly, the widths of both interval estimates increase whereas the coverage probabilities decrease. This can be happened due to the fact as we increase  $p$ , a majority of the experimental units are withdrawn from the experiment at the early phases.
4. When we compare the ML estimators with the corresponding Bayes estimators, we observe that the estimates obtained through Bayesian approach are far near to the actual value of the parameters than the classical approach. We observe the same phenomenon in the comparative study of confidence interval and Bayesian confidence interval, that is, the average width of the BCI is shorter than the corresponding confidence interval average width. This indicates that prior information about the parameters improves the estimates' efficiency.
5. The MLs and Bayes estimates obtained from the Type-I progressively hybrid censored samples under binomial removals taken from the considered dataset show the same results as we have observed from the simulation study.

Finally, we conclude that the suggested approach may apply to the physicians, doctors, engineers and others where this form of life-testing is required.

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**Conflicts of Interest** No potential conflict of interest was reported by the authors.

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